Electroweak Sudakov logarithms in the Coulomb gauge*

W. Beenakker [a†] and A. Werthenbach [a‡]

[a]Department of Physics, University of Durham, Durham DH1 3LE, U.K.

We describe a formalism for calculating electroweak Sudakov logarithms in the Coulomb gauge. This formalism is applicable to arbitrary electroweak processes. For illustration we focus on the specific reactions $e^+e^- \to f\bar{f}$ and $e^+e^- \to W_T^+W_T^-$, $W_L^+W_L^-$, which contain all the salient details of dealing with the various types of particles. We discuss an explicit two-loop calculation and have a critical look at the (non-)exponentiation and factorisation properties of the Sudakov logarithms in the Standard Model.

1. Introduction

At the next generation of linear e^+e^- colliders TeV-scale center-of-mass energies will be reached [1]. At these energies the effects arising from weak corrections are expected to be of the order of 10% or more [2,3], i.e. just as large as the well-known electromagnetic corrections. In order not to jeopardize any of the high-precision studies at these high-energy colliders, it is therefore indispensable to improve the theoretical understanding of the radiative corrections in the weak sector of the Standard Model (SM). In particular this will involve a careful analysis of effects beyond first order in the perturbative expansion in the (electromagnetic) coupling $\alpha = e^2/(4\pi)$.

The dominant source of radiative corrections at TeV-scale energies is given by the so-called Sudakov logarithms $\propto \alpha^n \log^{2n}(M^2/s)$, involving particle masses M well below the collider energy $\sqrt{s}$. These corrections have a unique origin, being related to collinear-soft singularities [4]. In a recent study we have investigated these Sudakov effects at two-loop level in the process $e^+e^- \to f\bar{f}$ [5], finding disagreement with three earlier studies [6]–[8]. Here we give a description of our formalism, which is based on the Coulomb gauge, and extend it to reactions with transverse and longitudinal (massive) gauge bosons in the final state. Especially the treatment of the longitudinal gauge bosons requires some special attention. In order to cover all the relevant features and subtleties of our method, it is sufficient to restrict the discussion to the virtual corrections. In fact, since the Sudakov logarithms originate from the exchange of soft, effectively on-shell gauge bosons, many of the features derived for these virtual corrections are intimately related to properties of the corresponding real-gauge-boson emission processes.

2. Electroweak Sudakov logarithms in the Coulomb gauge

In order to facilitate the calculation of the one- and two-loop Sudakov logarithms, we work in the Coulomb gauge for both massless and massive gauge bosons. The power of this gauge choice lies in the fact that the gauge-boson propagators become effectively transverse:

$$P^{\mu\nu}(k) = -\,i\,\frac{\vec{k}^{\,2}\,g^{\mu\nu} + k^\mu k^\nu - k^0\,(k^\mu n^\nu + n^\mu k^\nu)}{\vec{k}^{\,2}\,(k^2 - M^2 + i\epsilon)}$$
$$= \frac{-\,i}{k^2 - M^2 + i\epsilon}\Big[Q^{\mu\nu}(k) - \frac{k^2}{\vec{k}^{\,2}}\,n^\mu n^\nu\Big]. \quad (1)$$

Here k and M are the momentum and mass of the gauge boson, and n is the unit vector in the time direction, which enters by virtue of using the Coulomb gauge. The tensor

$$Q_{\mu\nu}(k) = -\sum_{\lambda=T} \varepsilon_\mu(k,\lambda)\,\varepsilon^*_\nu(k,\lambda) \quad (2)$$

is the polarization sum for the transverse helicity states. Therefore the gauge bosons are effectively

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transverse if $k^2 \ll \vec{k}^{\,2}$, which is the case for collinear gauge-boson emission at high energies ($k^2 \propto M^2$ and $\vec{k}^{\,2} \approx k_0^2 \gg M^2$). As a result of the effective transversality, the virtual Sudakov logarithms originating from vertex, box etc. corrections are suppressed (provided all kinematical invariants are of the same order as the CM energy squared). Hence, all virtual Sudakov logarithms are contained exclusively in the self-energies of the external on-shell particles [9,10] or the self-energies of any intermediate particle that happens to be effectively on-shell. The latter is, for instance, needed for the production of near-resonance unstable particles. The elegance of this method lies in its universal nature. Once all self-energies to all on-shell/on-resonance SM particles have been calculated, the prediction of the Sudakov form factor for an *arbitrary* electroweak process becomes trivial. The relevant self-energies for the calculation of the Sudakov logarithms involve the exchange of collinear-soft gauge bosons. The collinear-soft exchange of fermions, scalars and ghosts leads to suppressed contributions, since the propagators of these particles do not have the required pole structure.

2.1. The external wave-function factors

The calculation of the external wave-function factors in the Coulomb gauge is rather straightforward for the fermions. For massive gauge bosons, however, the mixing with the corresponding component of the Higgs doublet introduces an additional complication. For instance, consider the W boson and the would-be Goldstone boson ϕ. For a proper description of the on-shell W bosons we have to define the asymptotic $W^{\rm as}$ field in terms of the interacting W and ϕ fields:

$$W_\mu^{\pm,\,{\rm as}}(x) \;=\; Z_W^{-\frac{1}{2}}\, W_\mu^\pm(x) \,\pm\, i\,\delta Z_\phi\, \frac{\partial_\mu \phi^\pm(x)}{M_W} + \delta Z_n\, n_\mu n\cdot W^\pm(x) + \delta Z_\partial\, \frac{\partial_\mu\, \partial\cdot W^\pm(x)}{M_W^2}, \tag{3}$$

in such a way that the lowest-order ('free-field') propagators are retrieved for $W^{\rm as}$. This fixes the renormalization factors Z and δZ in terms of the self-energies of the interacting fields [10].

For transverse polarization states (T) the mixing with the ϕ field vanishes and the asympotic state effectively reduces to

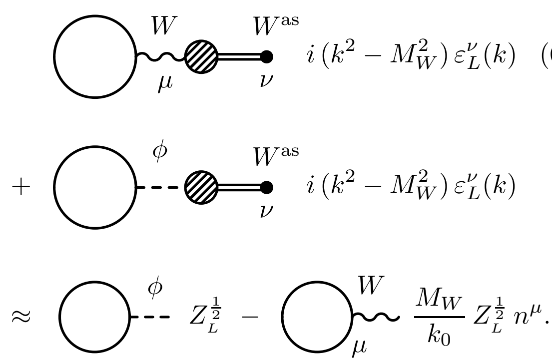
$$W_\mu^{\pm,\,{\rm as}}(x) \xrightarrow{T} Z_W^{-\frac{1}{2}}\, W_\mu^\pm(x), \tag{4}$$

since $\varepsilon_T(k)\cdot k = \varepsilon_T(k)\cdot n = 0$. The transverse wave-function factor Z_W is obtained from the purely transverse $Q_{\mu\nu}$ part of the Dyson-resummed gauge-boson self-energy [10]. The contribution of Sudakov logarithms now simply amounts to multiplying each external transverse gauge-boson line of the matrix element by the factor $Z_W^{\frac{1}{2}}$.

For longitudinal polarization states (L) we do have to deal with the W–ϕ mixing. In that case the relevant part of the asymptotic state is

$$W_\mu^{\pm,\,{\rm as}}(x) \xrightarrow{L} Z_W^{-\frac{1}{2}}\, W_\mu^\pm(x) + \delta Z_n\, n_\mu n\cdot W^\pm(x), \tag{5}$$

since $\varepsilon_L(k)\cdot k = 0$ and $\varepsilon_L(k)\cdot n \approx k_0/M_W$. In this case only the $n_\mu n_\nu$ part of the gauge-boson self-energy and the n_μ part of the W–ϕ mixing self-energy contribute. In the Sudakov limit these self-energies are related [10], resulting in the following identity for external longitudinal W bosons:



(6)

Here the last two diagrams are amputated at the external line. The longitudinal wave-function factor Z_L, defined as

$$Z_L^{-\frac{1}{2}} \equiv Z_W^{-\frac{1}{2}} + \delta Z_n, \tag{7}$$

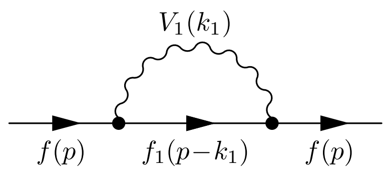
is obtained from the self-energy of the scalar ϕ field. The identity in Eq. (6) is in fact simply the Equivalence Theorem, stating that a non-vanishing matrix element for longitudinal W bosons at high energies is equivalent to the corresponding matrix element with the W bosons replaced by the would-be Goldstone bosons ϕ.

Hence, at high energies the would-be Goldstone bosons effectively become physical Goldstone bosons, at the expense of the longitudinal degrees of freedom of the massive gauge bosons. This is exactly what one would expect if the SM were to behave like an unbroken theory at high energies.

3. The Sudakov logarithms at one-loop

As an example we sketch the calculation of the one-loop Sudakov logarithms in the process $e^+e^- \to f\bar{f}$. Consider to this end the fermionic one-loop self-energy $\Sigma^{(1)}$, describing the emission of a gauge boson V_1 with loop-momentum k_1 and mass M_1 from a fermion f with momentum p and mass m_f:

$-i\,\Sigma^{(1)}(p,n) \;=$



In the high-energy limit the fermion mass in the numerator of the fermion propagator can be neglected. The self-energy $\Sigma^{(1)}$ then contains an odd number of γ-matrices, leading to the following natural decomposition in terms of the two possible structures $\not{p}$ and $\not{n}$:

$$\Sigma^{(1)}(p,n) \;\approx\; e^2\,\Gamma^2_{ff_1V_1}\Big[\not{n}\,\frac{p^2}{n\cdot p}\,\Sigma^{(1)}_n(n\cdot p,p^2) + \not{p}\,\Sigma^{(1)}_p(n\cdot p,p^2)\Big]. \quad (8)$$

The coupling factor $\Gamma_{ff_1V_1}$ is defined according to

$$\Gamma_{ff_1V_1} = V_{ff_1V_1} - \gamma_5\,A_{ff_1V_1}, \quad (9)$$

where $V_{ff_1V_1}$ and $A_{ff_1V_1}$ are the vector and axial-vector couplings of the fermion f to the exchanged gauge boson V_1.

The corresponding one-loop contribution to the external wave-function factor $Z_f = 1 + \delta Z_f$ can be obtained by means of the projection [5]

$$\delta Z_f^{(1)} \;=\; \frac{1}{2\,p_0}\,\bar{u}_f(p)\left\{\frac{\partial}{\partial p_0}\,\Sigma^{(1)}(p,n)\right\}u_f(p) \quad (10)$$
$$\approx\; -\int\frac{\mathrm{d}^4k_1}{(2\,\pi)^4}\,\frac{4\,e^2\,\Gamma^2_{ff_1V_1}\,p_\mu\,p_\nu\,P^{\mu\nu}(k_1,n)}{[(p-k_1)^2-m^2_{f_1}+i\epsilon]^2}\;.$$

In the last step we have exploited the fact that only collinear-soft gauge-boson momenta give rise to the Sudakov logarithms. Bearing in mind the effective transversality of the gauge-boson propagator $P^{\mu\nu}$ in that limit, the final expression exhibits the usual eikonal behaviour expected for the exchange of a soft gauge boson.

Having two canonical momenta at our disposal, i.e. p and n, we define the following Sudakov parametrisation of the gauge-boson loop-momentum k_1:

$$k_1 \;=\; v_1\,q + u_1\,\bar{q} + k_{1_\perp}\,, \quad (11)$$

with

$$p^\mu \equiv (E,\beta_f E,0,0), \qquad q^\mu = (E,E,0,0),$$
$$\bar{q}^\mu = (E,-E,0,0), \qquad k^\mu_{1_\perp} = (0,0,\vec{k}_{1_\perp}), \quad (12)$$

where $\beta_f = \sqrt{1-m_f^2/E^2}$ and $E \equiv \sqrt{s}/2$ are the velocity and energy of the external fermion f.

The v_1-integration is restricted to the interval $0 \le v_1 \le 1$, as a result of the requirement of having poles in both hemispheres of the complex u_1-plane. The residue is then taken in the lower hemisphere in the pole of the gauge-boson propagator: $s\,v_1\,u_1^{\mathrm{res}} = \vec{k}^{\,2}_{1_\perp} + M_1^2 \equiv s\,v_1\,y_1$. Finally, $\vec{k}^{\,2}_{1_\perp}$ is substituted by y_1, with the condition $\vec{k}^{\,2}_{1_\perp} \ge 0$ translating into $v_1\,y_1 \ge M_1^2/s$. The one-loop Sudakov contribution to δZ_f now reads

$$\delta Z_f^{(1)} \approx -\frac{\alpha}{\pi}\,\Gamma^2_{ff_1V_1}\int_0^1\frac{\mathrm{d}y_1}{y_1}\int_{y_1}^1\frac{\mathrm{d}z_1}{z_1}\;\mathcal{K}(y_1,z_1), \quad (13)$$

with the integration kernel $\mathcal{K}(y_1,z_1)$ given by

$$\mathcal{K}(y_1,z_1) \;=\; \Theta\Big(y_1 z_1 - \frac{M_1^2}{s}\Big)\,\Theta\Big(y_1 - \frac{m_f^2}{s}\,z_1\Big). \quad (14)$$

Here we introduced the energy variable $z_1 = v_1 + y_1$ and made use of the fact that only collinear-soft gauge-boson momenta are responsible for the quadratic large-logarithmic effects: $y_1,\, z_1 \ll 1$. As a result, the gauge boson inside the loop is effectively on-shell and transversely polarized. The θ-function containing the fermion mass m_f is needed for the exchange of photons only, regulating the collinear singularity at $y_1 = 0$. For the exchange of a massive gauge boson the mass M_1 will be the dominant collinear as well as infrared regulator. The final step of calculating the double (logarithmic) integral in Eq. (13) is now trivial.

The calculation of the Sudakov logarithms for other types of external particles proceeds in a similar way, using appropriate projection methods to bring the correction to the wave-function factor in an eikonal form [10].

Upon summation over the allowed gauge-boson exchanges, one obtains the following expression for the full one-loop Sudakov correction to the external wave-function factor for an arbitrary particle:

$$\delta Z^{(1)} = \left[\frac{C_2(R)}{\sin^2\theta_{\rm w}} + \left(\frac{Y}{2\cos\theta_{\rm w}}\right)^2\right]\mathrm{L}(M,M) + Q^2\Big[\mathrm{L}_\gamma(\lambda,m) - \mathrm{L}(M,M)\Big], \tag{15}$$

with $\theta_{\rm w}$ the weak mixing angle and

$$\mathrm{L}_\gamma(\lambda,M_1) = -\frac{\alpha}{4\pi}\left[\log^2\left(\frac{\lambda^2}{s}\right) - \log^2\left(\frac{\lambda^2}{M_1^2}\right)\right],$$
$$\mathrm{L}(M_1,M_2) = -\frac{\alpha}{4\pi}\log\left(\frac{M_1^2}{s}\right)\log\left(\frac{M_2^2}{s}\right). \tag{16}$$

Here m, eQ, Y and $C_2(R)$ are the mass, charge, hypercharge and $SU(2)$ Casimir operator of the external particle. So, $C_2(R) = C_F = 3/4$ for the fermions and longitudinal gauge bosons (read: Goldstone bosons), which are in the fundamental representation, and $C_2(R) = C_A = 2$ for transverse gauge bosons, which are in the adjoint representation. Finally, λ is the fictitious (infinitesimally small) mass of the photon needed for regularizing the infrared singularity at $z_1 = 0$. For the sake of calculating the leading Sudakov logarithms, the masses of the W and Z bosons can be represented by one generic mass scale M. Note that the terms proportional to Q^2 in Eq. (15) are the result of the mass gap between the photon and the weak bosons.

We have applied these one-loop Sudakov correction factors to the reactions $e^+e^- \to W_T^+W_T^-$, $W_L^+W_L^-$ and found perfect agreement with the high-energy approximation in Ref. [3], which confirms the afore-mentioned differences between transverse and longitudinal degrees of freedom.

4. Exponentiation revisited

In order to discuss the (non-)exponentiation and factorisation properties of the Sudakov logarithms in the SM, we now focus on the reaction $e^+e^- \to f\bar{f}$ at two-loop level. At two-loop accuracy one has to take into account the five generic sets of diagrams displayed in Fig. 1. Various cancellations are going to take place between all these diagrams. In unbroken theories like QED and QCD merely the so-called 'rainbow' diagrams of set (a) survive, and the resummation of the higher-order Sudakov effects amounts to an exponentiation of the one-loop corrections (see for instance Refs. [4,9,11,12]). The same holds if all gauge bosons of the theory would have a similar mass. The unique feature of the SM is that it is only partially broken, with the electromagnetic gauge group $U(1)_{\rm em} \neq U(1)_Y$ remaining unbroken. As such three of the four gauge bosons will acquire a mass, whereas the photon remains massless and will interact with the charged massive gauge bosons ($W^\pm$). As a result, the 'rainbow' diagrams are not going to be the only contributions that survive the gauge cancellations.

The generic two-loop contributions of Sudakov logarithms to δZ_f can be found in Ref. [5]. We merely note that several of the contributions involve a specific ordering in the energy variables z_i [in set (d) and part of set (e)] and/or the angular variables y_i [in sets (a),(d) and part of set (e)]. Note also that certain diagrams look possible at first sight, but are in fact forbidden as a result of the charged current interactions of the W bosons. For instance, in set (b) it is not possible to exchange two W bosons without reversing the fermion-number flow (given by the direction of the Dirac propagator lines). Adding up all possible contributions, we find for the full two-loop Sudakov correction factor for right- and left-handed fermions/antifermions

$$\delta Z^{(2)}_{f_L/\bar{f}_R} = \frac{1}{2}\left(\delta Z^{(1)}_{f_L}\right)^2 + \left(Q_f^2 - \frac{|Q_f|}{2\sin^2\theta_{\rm w}}\right)\Delta_f,$$
$$\delta Z^{(2)}_{f_R/\bar{f}_L} = \frac{1}{2}\left(\delta Z^{(1)}_{f_R}\right)^2 + Q_f^2\,\Delta_f, \tag{17}$$

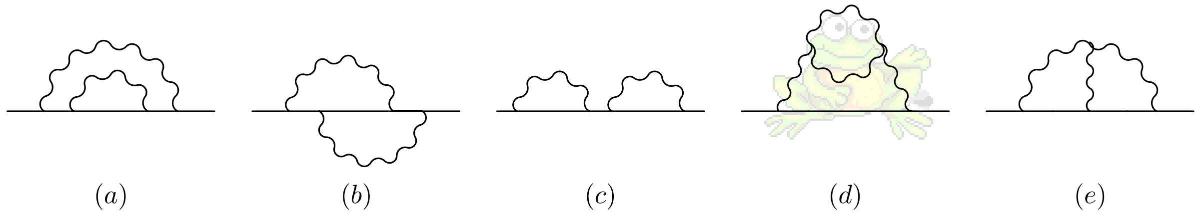


Figure 1. Generic two-loop fermionic self-energy diagrams contributing to the Sudakov logarithms

with

$$\Delta_f = \mathrm{L}(M,M)\left[\frac{4}{3}\,\mathrm{L}(M,m_f) - \mathrm{L}(M,M)\right] \qquad (18)$$

and $\mathrm{L}(M_1,M_2)$ as defined in Eq. (16). The expressions for transverse and longitudinal gauge bosons have a similar form [10].

From Eq. (17) we deduce the main statement of Ref. [5], namely that the virtual electroweak two-loop Sudakov correction factor is not obtained by a mere exponentiation of the one-loop Sudakov correction factor. Based on the explicit two-loop calculation we find non-exponentiating terms, originating from the mass gap between the *massless* photon and the *massive* Z boson in the neutral sector of the SM. We have checked that these extra terms vanish in leading-logarithmic approximation if *all* gauge bosons have the same (or roughly the same) mass. From Eq. (18) it is clear that the non-exponentiating terms are genuine quadratic large-logarithmic effects. They will not vanish if the fermion mass is of the order of the masses of the weak bosons or if the energy is taken to infinity, since in those cases $\Delta_f \;\to\; \mathrm{L}^2(M,M)/3$. Therefore, as far as the virtual Sudakov logarithms are concerned, the SM will never completely behave like an unbroken theory, even not if the energy becomes arbitrarily large. This is a consequence of the fact that the photon is massless, i.e. $m_f/\lambda \gg \sqrt{s}/M$.

We also note that, in adding up all the contributions, we find that the 'rainbow' diagrams of set (a) yield the usual exponentiating terms plus an extra term similar to the one found in Ref. [7], originating from the charged-current interactions. Whereas in Ref. [7] this extra term was interpreted as the source of non-exponentiation, we observe that it in fact cancels against a specific term originating from the triple gauge-boson diagrams of set (e). Similar (gauge) cancellations take place between the 'crossed rainbow' diagrams of set (b), the reducible diagrams of set (c), and another part of the triple gauge-boson diagrams of set (e). Finally, most of the left-over terms of set (e) get cancelled by the contributions from the gauge-boson self-energy ('frog') diagrams of set (d). The term proportional to $|Q_f|$ in Eq. (17) is the only surviving term of set (e), whereas the left–right symmetric term proportional to Q_f^2 originates from those diagrams of set (d) that involve neutral gauge bosons in the outer loop. The cancellation that usually takes place in unbroken gauge theories is upset by the fact that the on-shell poles for photons and Z bosons do *not* coincide, leading to different results for the on-shell residues. As was pointed out in Ref. [5], the energies of both the photon and the Z-boson are in the weak (soft-energy) domain, $z_1 \geq M/\sqrt{s}$, owing to the energy ordering. The observed difference is therefore caused entirely by the differences in the collinear domain induced by the mass gap. Based on this observation we conclude that the statements in Ref. [7] concerning the factorization and exponentiation properties of the Sudakov logarithms in the ultrasoft energy regime, $\lambda/\sqrt{s} \leq z_1 \leq M/\sqrt{s}$, are not contradicted by our analysis.

Comparing with the other two studies, we can make the following remarks. First of all, a treatment of pure weak gauge-boson effects without reference to the photonic interactions breaks gauge-invariance, since the photon has an explicit $SU(2)$ component. This holds even if the photon is treated fully inclusively as in Ref. [6]. Such a separation would require a very careful definition, for instance in terms of the typical energy regimes that govern the Sudakov effects of pure electromagnetic origin (ultrasoft energies) and collective electroweak origin (soft energies). Second, in contrast to Ref. [8] we do not observe the exponentiation of the virtual one-loop Sudakov logarithms. In the dispersive method of Ref. [8] it is assumed that QCD-like diagrammatic cancellations will take place, whereas we find that such cancellations can be upset by the fact that the on-shell poles for photons and Z bosons do not coincide. The latter might also have repercussions on the dispersive method itself, since in the diagrams of set (d) with one photon and one Z boson in the outer loop, it is not possible that both gauge-bosons are effectively on-shell simultaneously.

5. Conclusions and Outlook

We have presented a universal formalism for calculating high-energy electroweak Sudakov logarithms in the Coulomb gauge. In this special gauge all the relevant contributions, involving the exchange of collinear-soft gauge bosons, are contained in the self-energies of the external on-shell particles. In this context the treatment of longitudinal gauge bosons requires special care in view of their mixing with the would-be Goldstone bosons. After defining a proper asymptotic state, the equivalence of the longitudinal gauge-boson degrees of freedom and the Goldstone-boson degrees of freedom contained in the fundamental Higgs doublet becomes apparent. As such, the Sudakov logarithms for longitudinal gauge bosons are completely different from the ones for the transverse gauge bosons.

Our explicit one-loop results are in agreement with the calculations in the literature. At two-loop level, however, we do not observe a mere exponentiation of the one-loop results, in contrast to claims in the literature. The non-exponentiating terms in our two-loop result originate from the mass gap between the massless photon and the massive Z boson in the neutral sector of the SM. The cancellation that takes place in unbroken gauge theories, leading to exponentiation, is upset by the fact that the on-shell poles for photons and Z bosons do not coincide. We find that the corresponding non-exponentiating terms originate from the collinear domain and involve soft energies above the gauge-boson mass scale M.

From the explicit two-loop calculation we furthermore conclude that, as far as the virtual Sudakov logarithms are concerned, the SM will never completely behave like an unbroken theory at high energies. This is a consequence of the fact that the photon is strictly massless, being the gauge boson associated with the unbroken electromagnetic gauge group $U(1)_{\rm em}$.

A complementary study of the electroweak Sudakov logarithms for real collinear-soft gauge-boson emission processes is in progress.

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